FDTD Algorithm for Microstrip Antennas with Lossy Substrates Using Higher Order Schemes

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Higher order spatial schemes are applied to approximate the derivatives of the conducting Maxwell’s equations and an analytical study regarding the stability properties and the numerical dispersion of the N-order-in-space and second-order-in-time technique is considered. The hard-to-model effect of dielectric losses on microstrip antennas is investigated and their performance is demonstrated using a fourth-order-in-space and second-order-in-time, finite-difference time-domain scheme in three-dimensional calculations.

Keywords FDTD methods, higher order schemes, numerical dispersion, lossy media, microstrip antennas.

Introduction

Among existing techniques for solving electromagnetic problems, the finite-difference time-domain (FDTD) method (Taflove & Hagness, 2000) is a very competent second-order accurate, nondissipative, direct solution of the time-domain Maxwell’s equations on a staggered grid. Despite its efficiency, especially in implementing heterogeneous dielectrics and metal boundaries, the classical FDTD requires many grid points per wavelength to precisely simulate wave propagation. Errors from dispersion and anisotropy can be significant for large-scale or prolonged integration problems, unless the spatial discretization is extremely small. But mesh refinement can lead to prohibitive memory requirements and overwhelming computational cost.

Higher order schemes (Fang, 1989; Kantartzis & Tsiboukis, 2000; Georgakopoulos et al., 2002) seem to be a promising approach to minimize phase errors inherent in the original Yee method. Using such methods, coarser grids can be utilized compared to those
needed to accomplish equivalent levels of accuracy with the classical FDTD algorithm. Higher order FDTD schemes exhibit lower dispersion errors and enable the efficient simulation of electrically large practical problems. Fang (1989) first introduced fourth-order spatial and second- (and fourth-) order temporal accuracy algorithms. In Young, Gaitonde, and Shang (1997), an efficient compact scheme was merged with a Runge–Kutta integrator and in Hadi and Piket-May (1997) a modified version was introduced for modeling electrically large structures. In Georgakopoulos et al. (2001), higher order schemes were combined with subgridding techniques to handle any fine features of the structure, while Prokopidis and Tsiboukis (2003) presented an extension of the FDTD (2,4) technique to lossy dielectrics and its application to a two-dimensional waveguide problem.

In many applications involving aircraft and spacecraft where size and weight are constraints, low profile antennas may be required. Microstrip antennas (Balanis, 1997; Verma & Nasimuddin, 2003) are very thin and consequently rugged, lightweight, easy to design and mount, and economical to construct. Generally, the antenna element itself may be square, rectangular, round, etc., and may have more than one feed point. Unfortunately, many realistic applications involving microstrip patch antennas deal with lossy substrates. The effect of dielectric losses on several structures was not previously possible to analyze using higher order methods due to a lack of a stable and accurate scheme. The existing algorithm (Taflove & Hagness, 2000) for the simulation of electromagnetic waves in lossy media underlies the well-known drawbacks of the second-order scheme, such as the phase error and the numerical dispersion. Since such errors are controlled solely by the mesh size, to overcome these problems we have to refine the grid or use higher order schemes. We used the fourth-order spatial operator to derive a novel, stable, and accurate scheme. It is the objective of this paper to extend the higher order techniques to conducting media, study their dispersion characteristics, and derive stability conditions. We also present a novel three-dimensional higher order (2,4) method for the accurate simulation of electromagnetic problems with lossy media. Numerical results involve the analysis of a rectangular and a circular microstrip antenna with diverse values of substrate conductivities, via the calculation of $S$-parameters and input impedances.

### Higher Order FDTD Schemes in Lossy Dielectrics

Maxwell’s equations in a conductive and nonmagnetic medium of electric conductivity $\sigma$, with electric permittivity $\varepsilon = \varepsilon_r \varepsilon_0$ and magnetic permeability $\mu_0$, are expressed as

$$\frac{\partial \mathbf{E}}{\partial t} + \sigma \mathbf{E} = \nabla \times \mathbf{H},$$  \hspace{1cm} (1)  

$$\mu_0 \frac{\partial \mathbf{H}}{\partial t} = -\nabla \times \mathbf{E}. \hspace{1cm} (2)$$

One way to minimize the numerical dispersion of the original FDTD method is to improve the accuracy of the spatial difference equations used to approximate Maxwell’s curl equations. The most common approach to approximate spatial derivatives is through central operators. The central operator of $N$ order ($N$: even number) has the general form

$$\delta_{\beta} f_m = \sum_{l=1(l \text{ odd})}^{N-1} c_l^N (f_{m+l/2} - f_{m-l/2}), \hspace{1cm} (3)$$
where the coefficients \( c_N^l \) are calculated using Taylor series and are given in a closed form
\[
c_N^l = \frac{(-1)^{\frac{l}{2}}}{2 \left( \frac{l}{2} \right)^2} \frac{[(N - 1)!!]^2}{(N - 1 - l)!!(N - 1 + l)!!}, \quad l = 1, 3, 5, \ldots, N - 1, \tag{4}
\]
where the symbol !! implies the double factorial defined as
\[
n!! = \begin{cases} 
  n \cdot (n - 2) \cdots 5 \cdot 3 \cdot 1, & n > 0 \text{ odd,} \\
  n \cdot (n - 2) \cdots 6 \cdot 4 \cdot 2, & n > 0 \text{ even,} \\
  1, & n = -1, 0.
\end{cases} \tag{5}
\]
For example, we obtain \( c_2^1 = 1 \) (Yee’s scheme), \( c_4^1 = 9/8 \), and \( c_3^1 = -1/24 \) (Fang’s fourth-order scheme) and \( c_4^0 = 75/64 \), \( c_3^0 = -25/384 \), and \( c_6^0 = 3/640 \) (sixth-order scheme). Although a central operator of any order can be used to approximate the spatial derivatives of the Maxwell’s equations, it is very difficult to apply them in realistic nonhomogeneous problems due to their wide stencils. Thus, we will mainly concentrate on the fourth-order central operator, and we will discuss its proper modifications at the boundaries of the computational domain or near discontinuities.

We discretize (1), (2) using second-order temporal and fourth-order spatial approximations, while the conduction current term is approached using the time-average scheme (Taflove & Hagness, 2000). For the sake of brevity, only the \( E_x \)-component update equation is shown:
\[
E_{x,I+\frac{1}{2},J,K}^{n+1} = AE_{x,I+\frac{1}{2},J,K}^n - Bz \left( H_{y,I+\frac{1}{2},J,K-\frac{1}{2}}^{n+\frac{1}{2}} - 27H_{y,I+\frac{1}{2},J,K-\frac{1}{2}}^{n+\frac{1}{2}} + 27H_{y,I+\frac{1}{2},J,K+\frac{1}{2}}^{n+\frac{1}{2}} - H_{y,I+\frac{1}{2},J,K+\frac{3}{2}}^{n+\frac{1}{2}} \right) + By \left( H_{z,I+\frac{1}{2},J-\frac{1}{2},K}^{n+\frac{1}{2}} - 27H_{z,I+\frac{1}{2},J-\frac{1}{2},K}^{n+\frac{1}{2}} + 27H_{z,I+\frac{1}{2},J+\frac{1}{2},K}^{n+\frac{1}{2}} - H_{z,I+\frac{1}{2},J+\frac{3}{2},K}^{n+\frac{1}{2}} \right) \tag{6}
\]
where \( A, B_\beta \) are
\[
A = \frac{\varepsilon - 0.5\sigma \Delta t}{\varepsilon + 0.5\sigma \Delta t}, \quad B_\beta = \frac{\Delta t}{24\Delta \beta (\varepsilon + 0.5\sigma \Delta t)}, \tag{7}
\]
with \( \beta = y, z \). The classical Yee’s algorithm can be applied at all nodes in a bounded domain except at the first and last, where absorbing boundary conditions (ABCs) are to be imposed. However, higher order methods require numerical boundary conditions at the nodes next to an electric boundary node, which is the major drawback of the schemes with widened stencils. Various attempts have been made to provide proper boundary treatment. In Yefet and Petropoulos (2001), one-sided approximations of the derivatives and extrapolation have been proposed for the updating of the field in the neighborhood of the interface. A set of numerical boundary conditions based on the
Figure 1. The field components and the respective coefficients of the central (down) and the one-sided (up) stencil near the boundary.

Integral form of Maxwell’s equations were derived in Hwang and Cangellaris (2003). Herein, we use an extension of Yefet and Petropoulos (2001) in lossy media and we impose a fourth- and third-order accurate one-sided approximation at the two interior grid points (one electric and one magnetic) immediately next to the first and last electric field points of the bounded domain. In Figure 1 an electric component (black square) and the magnetic components (grey triangles) needed for its calculation are illustrated graphically. Notice that if the central (2,4) scheme was used, one component would be outside the computational domain. Using Taylor’s expansion theorem the third-order one-sided approximation of the derivative at \( k = 2 \) is

\[
\frac{dU}{dz} \bigg|_{2} = \frac{1}{24\Delta z} (-23U_{3/2} + 21U_{5/2} + 3U_{7/2} - U_{9/2}) + \frac{(\Delta z)^3}{24} \frac{d^4U}{dz^4} \bigg|_{2} + \cdots, \tag{8}
\]

with a leading error of order \( (\Delta z)^3 \). The weight coefficients of the previous scheme can be seen in Figure 1. Then by (8) the \( E_x \)-component one cell next to the \( k = 1 \) boundary is given by

\[
E^{n+1}_{x,l+\frac{1}{2},j,2} = AE^n_{x,l+\frac{1}{2},j,2} - B_x \left(-23H^{n+\frac{1}{2}}_{y,l+\frac{1}{2},j,3} + 21H^{n+\frac{1}{2}}_{y,l+\frac{1}{2},j,5} + 3H^{n+\frac{1}{2}}_{y,l+\frac{1}{2},j,7} - H^{n+\frac{1}{2}}_{y,l+\frac{1}{2},j,9} \right) + B_y \left(H^{n+\frac{1}{2}}_{z,l+\frac{1}{2},j-\frac{1}{2},2} - 27H^{n+\frac{1}{2}}_{z,l+\frac{1}{2},j-\frac{1}{2},2} + 27H^{n+\frac{1}{2}}_{z,l+\frac{1}{2},j+\frac{1}{2},2} - H^{n+\frac{1}{2}}_{z,l+\frac{1}{2},j+\frac{3}{2},2} \right). \tag{9}
\]

In dielectric interfaces we approximate electric and magnetic fields by using fifth-order extrapolation. Unfortunately, one-sided approximations can yield instabilities in many practical three-dimensional problems, where perfectly electric conductor (PEC) planes are not infinite in extent (Georgakopoulos et al., 2002). Generally, modeling of discontinuities and dielectric interfaces in higher order FDTD lattices remains an active research field.
Higher Order FDTD Stability Analysis

The electric field is assumed to be of the following single frequency form:

$$E_{n, I, J, K}^n = E_0 e^{j(\omega_n \Delta t + \hat{k}_x I \Delta x + \hat{k}_y J \Delta y + \hat{k}_z K \Delta z)}$$ (10)

where $E_0$ is a complex amplitude, indexes $I$, $J$, $K$ denote the position of the nodes in the FDTD mesh, $\Delta x$, $\Delta y$, $\Delta z$ are the sizes of the discretization cell, and $\hat{k}_x$, $\hat{k}_y$, $\hat{k}_z$ are the numerical wavenumbers of the discrete modes in each Cartesian direction, which for a lossy medium are generally complex numbers. A tilde is used to distinguish the numerical and physical quantities.

We consider the wave equation of an electric field component instead of the coupled set of Maxwell’s equations (1), (2) in a source-free, homogeneous, and lossy medium

$$\left( \frac{1}{\mu_0 \varepsilon} \sum_{\beta=x,y,z} \delta_\beta^2 \frac{\partial^2}{\partial \beta^2} \varepsilon (\Delta \beta)^2 - \frac{1}{\tau} \frac{\partial}{\partial t} \frac{\partial^2}{\partial \tau^2} \right) E(r, t) = 0,$$ (11)

where $\tau = \varepsilon / \sigma$ is the medium’s relaxation time. We apply the FDTD (2,4) scheme to the previous equation with the remark that the conduction term is discretized using the time averaging approach. The resulting difference equation is

$$\left( \frac{1}{\mu_0 \varepsilon} \sum_{\beta=x,y,z} \delta_\beta^2 \frac{\delta_\beta^2}{(\Delta \beta)^2} \varepsilon (\Delta \beta)^2 - \frac{1}{\tau} \frac{\delta_t}{\Delta t} \frac{\delta_\beta^2}{(\Delta \beta)^2} \right) E_{I, J, K}^n = 0,$$ (12)

with $\Delta t$ being the temporal step size, $\delta_t$ the usual central difference operator, and $\Delta \beta$, $\delta_\beta$ the spatial step size and the fourth-order difference spatial operator of coordinate $\beta$, respectively. We incorporate the finite average operator $\mu_t$ to approximate the lossy term defined as

$$\mu_t f(n \Delta t) \equiv \mu_t f^n = \frac{f^{n+\frac{1}{2}} + f^{n-\frac{1}{2}}}{2}.$$ (13)

The central difference operator with respect to time is defined as

$$\delta_t f(n \Delta t) \equiv \delta_t f^n = f^{n+\frac{1}{2}} - f^{n-\frac{1}{2}},$$ (14)

and consequently,

$$\delta_\beta^2 f^n \equiv \delta_\beta [\delta_\beta f^n] = f^{n+1} - 2 f^n + f^{n-1}.$$ (15)

The fourth-order spatial operator is denoted as

$$\delta_\beta f_m = \frac{1}{24} (f_{m+\frac{3}{2}} - 27 f_{m-\frac{1}{2}} + 27 f_{m+\frac{1}{2}} - f_{m+\frac{5}{2}}),$$ (16)

where the index $m$ corresponds to $\beta$ coordinate, and thus

$$\delta_\beta^2 f_m \equiv \delta_\beta [\delta_\beta f_m] = \frac{1}{576} (f_{m-3} - 54 f_{m-2} + 783 f_{m-1} - 1460 f_m + 783 f_{m+1} - 54 f_{m+2} + f_{m+3}).$$ (17)
We substitute the previous operators in the discretized wave equation (12) and we take the $Z$-transform, with respect to time, of the difference equation, bearing in mind the property $Z\{f^{n-k}\} = Z^{-k}F(Z)$. We obtain the following second-degree polynomial in the $Z$-domain:

$$\left(1 + \frac{\Delta t}{2\tau}\right)Z^2 + (4\rho^2 - 2)Z + \left(1 - \frac{\Delta t}{2\tau}\right) = 0,$$

(18)

where

$$\rho^2 = \frac{(\Delta t)^2}{\mu_0\varepsilon} \sum_{\beta=x,y,z} \frac{1}{(\Delta\beta)^2} \left[\frac{9}{8} \sin \left(\frac{\tilde{k}_{\beta}\Delta\beta}{2}\right) - \frac{1}{24} \sin \left(\frac{3\tilde{k}_{\beta}\Delta\beta}{2}\right)\right]^2,$$

(19)

and $\tilde{k}_{\beta}$ is the numerical wavenumber in the $\beta$ direction. Since $\tilde{k}_{\beta}$ are complex numbers and $\sin(\imath z) = \imath \sinh(z)$ for any complex number $z$, we get the following alternative expression:

$$\sin \left(\frac{\tilde{k}_{\beta}\Delta\beta}{2}\right) = \imath \sinh \left(-\imath \frac{\tilde{k}_{\beta}\Delta\beta}{2}\right).$$

(20)

A similar expression can be obtained for $\sin(3\tilde{k}_{\beta}\Delta\beta/2)$. The previous methodology can be used to yield the stability polynomial for the (2,6) scheme. In this case, the parameter $\rho^2$ will be

$$\rho^2 = \frac{(\Delta t)^2}{\mu_0\varepsilon} \sum_{\beta=x,y,z} \frac{1}{(\Delta\beta)^2} \left[\frac{75}{64} \sin \left(\frac{\tilde{k}_{\beta}\Delta\beta}{2}\right) - \frac{25}{384} \sin \left(\frac{3\tilde{k}_{\beta}\Delta\beta}{2}\right) + \frac{3}{640} \sin \left(\frac{5\tilde{k}_{\beta}\Delta\beta}{2}\right)\right]^2.$$

(21)

We investigate the stability of the (2,4) scheme by means of a combination of the von Neumann method with the Routh–Hurwitz criterion (Pereda et al., 2001). All the roots of the previous polynomial must be inside or on the unit circle in the $Z$-plane, i.e., $|Z| \leq 1$, for the proposed scheme to be stable. We use the methodology described in Pereda et al. (2001) and apply the following bilinear transformation:

$$Z = \frac{r + 1}{r - 1},$$

(22)

to (18) and obtain the following stability polynomial in the $r$-plane:

$$2\rho^2 r^2 + \frac{\Delta t}{\tau} r + 2(1 - \rho^2) = 0.$$

(23)

The Routh table for this polynomial is

\[
\begin{array}{cc}
2\rho^2 & 2(1 - \rho^2) \\
\frac{\Delta t}{\tau} & 0 \\
2(1 - \rho^2) & \\
\end{array}
\]
Forcing the entries of the first column of the above table to be nonnegative quantities, we obtain the stability condition \( \rho^2 \leq 1 \). Applying the previous inequality to (19), we get

\[
\Delta t \leq \sqrt{\mu_0 \varepsilon} \left\{ \sum_{\beta=x,y,z} \frac{1}{(\Delta \beta)^2} \left[ \frac{9}{8} \sin \left( \frac{\tilde{k}_\beta \Delta \beta}{2} \right) - \frac{1}{24} \sin \left( \frac{3\tilde{k}_\beta \Delta \beta}{2} \right) \right] \right\}^{-1/2},
\]

and for practical calculations the worst case is taken, i.e., \( \sin(\tilde{k}_\beta \Delta \beta/2) = 1 \) and \( \sin(3\tilde{k}_\beta \Delta \beta/2) = -1 \), so that the following more restrictive stability condition is derived:

\[
\Delta t \leq \frac{6}{7} \sqrt{\mu_0 \varepsilon} \left[ \sum_{\beta=x,y,z} \frac{1}{(\Delta \beta)^2} \right]^{-1/2}.
\]

The stability condition is exactly the same as for the lossless media (Fang, 1989), an observation that is very convenient for our simulations. As a consequence, there is no need to decrease the time step of our FDTD calculations if medium conductivities are present. Similarly, for the (2,6) scheme we have

\[
\Delta t \leq \frac{120}{149} \sqrt{\mu_0 \varepsilon} \left[ \sum_{\beta=x,y,z} \frac{1}{(\Delta \beta)^2} \right]^{-1/2}.
\]

For a uniform grid \( (\Delta x = \Delta y = \Delta z = \Delta \beta) \) we can define the Courant number \( Q \) as

\[
Q = \frac{c_0 \Delta t}{\Delta \beta},
\]

where \( c_0 = 1/\sqrt{\mu_0 \varepsilon} \) is the velocity of light in free space. For three-dimensional FDTD (2,4) simulations \( Q_{2,4} \leq 6/(7\sqrt{3}) \cong 0.495 \), while for three-dimensional FDTD (2,2) \( Q_{2,2} \leq 1/\sqrt{3} \cong 0.577 \) (in free space). In a medium with relative permittivity \( \varepsilon_r \), we have \( Q_{2,4} \leq 6\sqrt{\varepsilon_r}/(7\sqrt{3}) \) and \( Q_{2,2} \leq \sqrt{\varepsilon_r}/\sqrt{3} \).

**Dispersion Analysis of the Higher Order Schemes**

Numerical dispersion is a nonphysical effect always present in the FDTD method that causes distortion of waveforms and cumulative phase error. This nonphysical effect can be estimated through a numerical dispersion relation (Taflove & Hagness, 2000).

The expression of the numerical dispersion of the (2,4) scheme can easily be found by letting \( Z = \exp(j\omega \Delta t) \) in the polynomial (18):

\[
\sin^2(\omega \Delta t/2) \left[ \frac{\varepsilon_r - j \sigma \Delta t}{2\epsilon_0 \tan(\omega \Delta t/2)} \right] = \sum_{\beta=x,y,z} \frac{1}{(\Delta \beta)^2} \left[ \frac{9}{8} \sin \left( \frac{\tilde{k}_\beta \Delta \beta}{2} \right) - \frac{1}{24} \sin \left( \frac{3\tilde{k}_\beta \Delta \beta}{2} \right) \right]^2,
\]

where \( \tilde{k}_x, \tilde{k}_y, \) and \( \tilde{k}_z \) are \( x, y, \) and \( z \) components of the numerical wave vector \( \tilde{k} \), respectively, given by

\[
\tilde{k}_x = \tilde{k} \sin \theta \cos \phi, \quad \tilde{k}_y = \tilde{k} \sin \theta \sin \phi, \quad \tilde{k}_z = \tilde{k} \cos \theta.
\]
with $\theta, \phi$ being the spherical coordinates of the wave direction. The numerical dispersion for the (2,6) scheme is

$$\begin{align*}
\sin^2\left(\frac{\omega \Delta t}{2}\right) \left[ \varepsilon_r - \frac{j \sigma \Delta t}{2\varepsilon_0 \tan(\omega \Delta t/2)} \right] \\
= \sum_{\beta=x,y,z} \frac{1}{(\Delta \beta)^2} \left[ 75 \sin \left( \frac{k_{\beta} \Delta \beta}{2} \right) - \frac{25}{384} \sin \left( \frac{3k_{\beta} \Delta \beta}{2} \right) + \frac{3}{640} \sin \left( \frac{5k_{\beta} \Delta \beta}{2} \right) \right]^2
\end{align*}$$

(30)

and can be generalized to the general $(2, N)$ scheme as

$$\begin{align*}
\sin^2\left(\frac{\omega \Delta t}{2}\right) \left[ \varepsilon_r - \frac{j \sigma \Delta t}{2\varepsilon_0 \tan(\omega \Delta t/2)} \right] \\
= \sum_{\beta=x,y,z} \frac{1}{(\Delta \beta)^2} \left[ \sum_{l=1(l \text{ odd})}^{N-1} c_l^N \sin \left( \frac{l k_{\beta} \Delta \beta}{2} \right) \right]^2
\end{align*}$$

(31)

The wavenumber $k$ and the wavelength $\lambda$ in the physical world are

$$k = \frac{2\pi}{\lambda} = \omega \sqrt{\frac{\mu_0 \varepsilon_r}{2} \left[ 1 + \left( \frac{\sigma}{\omega \varepsilon_r} \right)^2 \right]}$$

(32)

The dispersion relation of the FDTD (2,2) scheme in lossy dielectrics is found in Pereda et al. (1998). To examine the accuracy of the scheme, we take the ratio of the numerical phase velocity $\tilde{c}$ to the exact $c$:

$$\frac{\tilde{c}}{c} = \frac{\omega / \text{Re}[k]}{\omega / k} = \frac{k}{\text{Re}[k]}$$

(33)

where $k$ is calculated through (32) and $\tilde{k}$ is computed using (28) by applying Newton’s iterative method. The numerical dispersion of FDTD (2,4) scheme is investigated to bring forward some salient features. We perform a comparison of the proposed scheme with the FDTD (2,2) method in a medium with $\sigma = 0.2$ S/m, $\varepsilon_r = 1$, and a uniform lattice with $\Delta \beta = 0.1$ m and Courant number $Q = 0.3$. Figure 2 shows the normalized phase velocity for three different values of grid resolution $N_{\lambda} = \lambda / \Delta \beta$ and for various directions of propagation $\phi (\theta = \pi/2)$. Notice that the phase speed error for the fourth-order scheme with $N_{\lambda} = 3.79$ is smaller even than that of the second-order with $N_{\lambda} = 5.48$ for all angles of wave propagation.

Moreover, the following definition of the phase error:

$$\Phi(\omega \Delta t) = \left| \frac{k(\omega) - \tilde{k}(\omega \Delta t)}{k(\omega)} \right|$$

(34)

and the ratio of the time step to the medium’s relaxation time $\dot{h} = \Delta t / \tau$ is considered. Figure 3 depicts the phase error of the classical FDTD and the proposed (2,4) scheme for a uniform three-dimensional grid with $Q = 0.4$, $\sigma = 0.1$ S/m, and $\varepsilon_r = 1$. The spatial
Figure 2. Polar diagram of the ratio of the numerical phase velocity to the physical velocity $\tilde{c}/c$ for three values of $N_A$: 3.79, 5.48, 7.84 for the proposed FDTD (2,4) and FDTD (2,2).

increment $\Delta \beta$ is solely determined by (27), while the temporal step $\Delta t$ through the relation $\Delta t = h \tau$, for diverse values of $h$. We observe that to achieve a phase error lower than $10^{-6}$ with FDTD (2,2), we need to use ten times smaller time steps than FDTD (2,4). Therefore, using higher order schemes, we can reduce the total number of time steps (and thus the execution time) of FDTD simulations while maintaining the same error.

The FDTD (2,2) scheme requires that the Courant number $Q_{2,2}$ be the maximum allowed for stability in order to introduce the least phase error (Taflove & Hagness, 2000). Once the spatial resolution is set according to minimum wavelength of the excitation, one has to reduce the Courant number, and as a result the phase error introduced by the scheme increases. However, the (2,4) scheme is stable for $Q_{2,4} \leq 6(7/\sqrt{3}) \approx 0.495$ and operates well for small values of $Q$ and does not suffer from the same phase error degradation of classical FDTD as the Courant number is decreased. Therefore, the novel scheme diminishes dispersion and anisotropy errors and seems to be very suitable for realistic and computational costly FDTD simulations.

Numerical Results

The effect of dielectric losses in planar microstrip structures has been previously evaluated through FDTD (2,2) schemes in Wittwer and Ziolkowski (2001), but only for rectangu-
lar microstrip antennas. Here, we examine the importance of substrate conductivity in rectangular and circular microstrip antennas using the proposed higher order technique. Rogers, the manufacturer of the substrate material (Duroid 5880), cites a loss tangent of \( \tan \delta = 9.0 \times 10^{-4} \) at a frequency of 10 GHz, which is equivalent to a \( \sigma = 1.1 \times 10^{-3} \) S/m conductivity value and a relative electric permittivity \( \varepsilon_r \) of 2.2. We model the substrate by changing the conductivity value \( \sigma \). We will refer to the case of zero conductivity as the “lossless” case and the case of \( \sigma = 0.01 \) S/m and \( \sigma = 0.1 \) S/m as “lossy” and “very lossy” cases, respectively.

To illustrate the behavior of the proposed FDTD (2,4) scheme we calculate the scattering parameter \( S_{11} \) and the input impedance \( Z_{in} = R_{in} + jX_{in} \) for a rectangular and a circular patch antenna. The geometry for the rectangular microstrip antenna is given in Figure 4a. The simulation region is \( 69 \times 80 \times 18 \) cells in \( x-, y-, \) and \( z- \)directions, respectively. The cell sizes are \( \Delta x = 0.389 \) mm, \( \Delta y = 0.4 \) mm, \( \Delta z = 0.265 \) mm, and the substrate is \( 3\Delta z \) thick. The rectangular microstrip is fed through a transmission line. The microstrip transmission line is modeled as an infinitely thin conductor in the \( z = 4\Delta z \) plane and \( 6\Delta x \) cells thick. The time step \( \Delta t \) resulting from (25) is 49.431 psec while the simulation lasts 4500 time steps. A source with Gaussian profile is placed ten cells away from edge of the microstrip line, as shown in Figure 4a. We have used the feeding model described in Luebbers and Langdon (1996) with internal resistance \( R_s = 50 \) \( \Omega \) in order for the electric fields to decay quickly.

Using the results from the FDTD simulations, the source voltages and currents are multiplied with a four-term Blackman–Harris window (Harris, 1978) and Fourier-transformed using the FFT, and then the input impedance \( Z_{in} \) at the feed location is

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**Figure 3.** Phase error for the lossy medium versus circular frequency for FDTD (2,2) and FDTD (2,4), \( Q = 0.4 \) and various values of \( h \).
The calculated scattering $S_{11}$ parameter for the three conductivity values is shown in Figure 5. Figure 6 depicts the input impedance of the rectangular antenna under consideration for the lossless, lossy, and very lossy cases. As one could expect, the high-$Q$ of the patch structure is seriously disturbed by the conductivity value of the substrate. However, such disturbances do not appear in the results of the $S_{11}$ parameter. This remark indicates that only scattering parameters do not give a complete picture of the antenna operation. These observations confirm the conclusions of Wittwer and Ziolkowski (2001).

The circular microstrip antenna examined here has only one feed point and is fed from the back. The feed point is located a short distance out from the center $O$, and again the feeding model of Luebbers and Langdon (1996) is applied. The dimensions of the circular microstrip antenna are given in Figure 4b. The domain is discretized into $80 \times 80 \times 18$ cells with $\Delta x = \Delta y = 0.4$ mm, $\Delta z = 0.2$ mm, and $\Delta t = 42.019$ psec, and the substrate is $3 \Delta z$ thick. The radius $a$ of the antenna is 12 mm and the circular disk is modeled using staircase approximations. The impedance at the center of the disk is zero, while at the outer edge it is very high (hundreds of ohms), so the distance from the center of the feed point is generally determined in order to achieve a good match with the coax cable. In our case this distance is 4 mm. The effect of the three values of conductivity on the calculated $S_{11}$ parameter is shown in Figure 7. The differences between the lossless and lossy cases are generally small, with no significant deviations in the nulls. However, the impedance calculations are more sensitive than the scattering parameters, as we can see in Figure 8. For the circular antenna the lowest order resonant frequency of the TM$_{110}$ mode is given by Balanis (1997)

$$ (f_r)_{110} = \frac{1.841c_0}{2\pi a \sqrt{\varepsilon_r}}. $$

(35)
Figure 5. $|S_{11}|$ for the rectangular microstrip antenna ($\sigma = 0, 0.01, 0.1$ S/m).

Figure 6. Input impedance for the rectangular microstrip antenna ($\sigma = 0, 0.01, 0.1$ S/m).
Figure 7. $|S_{11}|$ for the circular microstrip antenna ($\sigma = 0, 0.01, 0.1 \text{ S/m}$).

Figure 8. Input impedance for the circular microstrip antenna ($\sigma = 0, 0.01, 0.1 \text{ S/m}$).
For the examined problem, \( a = 12 \text{ mm} \) and \( \varepsilon_r = 2.2, \) so \( (f_r)_{110} = 4.94 \text{ GHz}, \) which is very close to the first null of Figure 7. In all simulations the computational domain was truncated with an eight-cell sponge layer (Petropoulos et al., 1998) indicating its effectiveness as an ABC for higher order FDTD lattices modeling lossy media.

**Conclusions**

In this paper, a novel and easy-to-implement FDTD (2,4) scheme for the treatment of lossy dielectrics has been demonstrated. A systematic methodology for the extraction of the stability condition and the dispersion relation of the general \((2, N)\) scheme was also presented. We have shown, through dispersion analysis, that via the proposed technique a coarser grid than the ordinary FDTD one can be evaluated, while maintaining the same accuracy. Moreover, the proposed scheme was implemented into three-dimensional FDTD codes. Numerical verifications of a rectangular and, for the first time, a circular microstrip antenna proved that the proposed technique can be used in the analysis and design of microstrip structures with lossy substrates. Furthermore, the proposed formulation can be effortlessly combined with the hybrid fourth-order FDTD method to simulate wave propagation over electrically large structures. Finally, we have also confirmed the utility of the higher order reflectionless sponge layers for the truncation of domains with lossy media.

**References**


