A General Form of Perfectly Matched Layers for Three-Dimensional Problems of Acoustic Scattering in Lossless and Lossy Fluid Media

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Abstract—The concept of perfectly matched layer (PML) has been proven very effective in absorbing electromagnetic waves in lossless media. An extension of this method and a complete three-dimensional (3-D) scheme properly suited for finite-difference, time-domain (FDTD) modeling of acoustic propagation and scattering in unbounded problems are presented in this paper. This generalized PML is constructed in such a way that it performs significant absorption of traveling waves in acoustics for both lossless and lossy media. Theoretically, no reflections occur when propagating waves encounter the lossy medium-PML interface, no matter what the angle of incidence is, introducing at the same time the possibility for further wave attenuation via the stretched coordinates idea. Numerical results support the suggested PML theory as well as reveal the proper modifications, which lead to the achievement of the optimum absorbing-boundary condition.

I. INTRODUCTION

DURING the recent years, the FDTD method has been widely used for the proper simulation of numerous problems not only in electromagnetics but also in acoustics. In fact, Yee’s scheme [1], [2] was found invaluable in combination with the appropriate absorbing boundary conditions (ABCs). Due to the enforcement of such conditions at the truncated computational domain, FDTD is enabled to treat all open-boundary problems that need to be computationally limited. That is, the propagating waves can be absorbed without reflections after the proper enforcement of an ABC at the truncation boundary; thus the FDTD simulation of the unbounded problem will be feasible.

Many techniques have been proposed in order to accomplish the best possible absorption. In this context, Berenger suggested the perfectly matched layer (PML), an artificial anisotropic lossy material that encloses the space of interest, and it has been proved to be the most efficient absorbing boundary conditions [3], [4]. The method was also found useful in acoustics, and it was used in many problems with similar results of high efficiency [5]–[7].

Despite the prior advantages, the PML was originally designed to handle only lossless propagation media. As a consequence, it was practically ineffective in all cases concerning losses. This is of great importance in acoustics as losses usually must be considered in most real media of interest. There have been efforts to improve PML and adjust it to absorb traveling waves in lossy materials. Among them, a generalized form, which relies on the assumption of additional attenuation coefficients in a stretched coordinate system, has been presented in electromagnetics [8] and for the first time in acoustics [9].

In this paper, a FDTD-PML technique is implemented in acoustic scattering to accomplish the desirable dissipation of the propagating waves at the truncation boundaries. The acoustic equations are considered, and the complete analysis of the proposed method for theoretically zero reflections is presented. The reflection factor of an incident plane wave striking upon a lossy medium-PML interface is imposed to be zero for all frequencies and angles of incidence in order to derive the conditions that must be fulfilled and consequently build up our absorber. At the same time, using the stretched coordinates idea, additional attenuation parameters were introduced to further dissipate the outgoing waves. A detailed formula, suitable for FDTD implementation, is simply and fully presented for the 3-D case. The proper general PML equations are derived in such a way that can be easily enforced in both lossless and lossy media after slight modification of its parameters, overcoming the main drawback of the original PML form. This is verified by several numerical experiments, which show the perfect matching of the PML layers as well as the high efficiency in the absorption and further attenuation of traveling waves.

II. ACOUSTIC EQUATIONS IN STRETCHED COORDINATES

In the general case of a homogeneous, lossy, fluid medium, the pressure-velocity acoustic equations, which express the well-known Newton’s law of motion and the equation of continuity, respectively, are:

\[ \nabla_s p = -\rho \frac{\partial \bar{u}}{\partial t} - a^* \bar{u}, \]

\[ \nabla_s \cdot \bar{u} = -\kappa \frac{\partial p}{\partial t} - ap, \]

where \( p \) is the pressure and \( \bar{u} \) is the vector of the oscillating or particle velocity in the fluid, respectively. The coefficients \( \rho, \kappa, \) and \( a \) symbolize the mass density, the compressibility, and the compressibility attenuation of the medium, respectively. The coefficient \( a^* \) is a nonphysical

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attenuation parameter, which is zero for all real acoustic media and used here for purposes of equation symmetry as well as for an easy reference to the original PML form which suggests that [3], [5]. The velocity of the sound in the medium is given by the expression \( c = 1/\sqrt{\rho \mu} \). The tensor \( \nabla_s \) indicates that a stretched coordinate system has been assumed and in rectangular coordinates takes the following form:

\[
\nabla_s = \hat{x} \frac{1}{s_x} \frac{\partial}{\partial x} + \hat{y} \frac{1}{s_y} \frac{\partial}{\partial y} + \hat{z} \frac{1}{s_z} \frac{\partial}{\partial z},
\]

where \( s_x, s_y, \) and \( s_z \) are the coordinate-stretching variables that stretch the coordinates in the \( x, y, z \) directions, respectively, with \( \hat{x}, \hat{y}, \hat{z} \) being the corresponding unit vectors. It must be noted that the stretched coordinates, as well as the whole PML absorber, are not associated with any physical material or process of absorption. Actually, it is just an abstraction that can be considered only as a mathematical way to introduce additional degrees of freedom regarding the further attenuation of the waves inside the artificial absorbing medium that is going to enclose the real lossy medium of interest [8], [10]. For the latter of course, \( s_x = s_y = s_z = 1 \) as the stretched coordinates refer only to the artificial PML absorber.

In the frequency domain, for acoustic waves of angular frequency \( \omega \), the above equations can be written as:

\[
\nabla_s p = (-j \omega p - \alpha^*) \vec{u} = -j \omega \rho' \vec{u},
\]

\[
\nabla_s \cdot \vec{u} = (-j \omega \kappa - \alpha) p = -j \omega \kappa' p,
\]

where:

\[
\rho' = \rho + \frac{\alpha^*}{j \omega}, \quad \kappa' = \kappa + \frac{\alpha}{j \omega}.
\]

Plane waves propagating in the acoustic medium are considered, which can be expressed as:

\[
p = p_0 e^{-j \beta \cdot \vec{r}}, \quad \vec{u} = \vec{u}_0 e^{-j \beta \cdot \vec{r}},
\]

with \( \beta = \beta_x \hat{x} + \beta_y \hat{y} + \beta_z \hat{z} \).

Hence,

\[
\nabla_s p = -j \beta_s p,
\]

\[
\nabla_s \cdot \vec{u} = -j \vec{\beta}_s \cdot \vec{u},
\]

where:

\[
\vec{\beta}_s = \frac{\beta_x}{s_x} \hat{x} + \frac{\beta_y}{s_y} \hat{y} + \frac{\beta_z}{s_z} \hat{z}.
\]

Multiplying (9) by \( \vec{\beta}_s \) and using (10), the dispersion relation is obtained:

\[
\vec{\beta}_s \cdot \vec{\beta}_s p = \omega \rho' \vec{\beta}_s \cdot \vec{u} = \omega^2 \kappa' \rho' p = 0,
\]

\[
\beta^2 = \omega^2 \kappa' \rho' = \frac{\beta_x^2}{s_x^2} + \frac{\beta_y^2}{s_y^2} + \frac{\beta_z^2}{s_z^2},
\]

where \( \beta \) is a complex number whose real and imaginary parts stand for the propagation and loss term of the wave, respectively.

The main idea of the original form of the PML absorbing boundary, which works only for lossless media, is to split the acoustic pressure into three additive terms \( p_x, p_y, \) and \( p_z \) and assume the compressibility attenuation \( \alpha \) diagonally anisotropic: \( \alpha = \text{diag}(\alpha_x, \alpha_y, \alpha_z) \) [3], [5]. This artificial medium encloses a lossless medium of interest, for which \( \alpha = 0 \). In this work, however, we consider a lossy medium with \( \alpha_1 \) the compressibility attenuation to be enclosed by a modified PML absorber having \( \alpha_2 \) as the compressibility attenuation. That is, the acoustic equations for both the real medium of the problem and the artificial, modified PML medium can take the form of:

\[
\frac{\partial p}{\partial \xi} = -j \omega \rho_1 u_\xi, (\xi = x \text{ or } y \text{ or } z),
\]

\[
\frac{\partial u_x}{\partial \xi} + \frac{\partial u_y}{\partial \eta} + \frac{\partial u_z}{\partial \zeta} = -j \omega \kappa_1' p,
\]

and

\[
\frac{1}{s_\xi} \frac{\partial (p_x + p_y + p_z)}{\partial \xi} = -j \omega \rho_2' u_\xi, (\xi = x \text{ or } y \text{ or } z),
\]

\[
\frac{1}{s_\xi} \frac{\partial u_\xi}{\partial \xi} = -j \omega \kappa_2' p_\xi, (\xi = x \text{ or } y \text{ or } z).
\]

If \( \alpha_1 = \alpha_2 = 0 \) (\( \alpha_1^*, \alpha_2^* \) are always zero), the analysis is this for an ordinary lossless acoustic medium.

### III. Wave Incidence at the Boundary and Perfectly Matching

In order to achieve a theoretically perfect absorption of the incident waves in the region of the proposed PML absorber, the parameters \( \kappa_2, \rho_2, \alpha_2, \alpha_2^* \), and \( s_x, s_y, \) and \( s_z \), which have been considered in the above defined PML equations, should be determined in such a way that the interface will be reflectionless.

In an oblique incidence of angle \( \theta_1 \), as shown in Fig. 1, let us assume that the incident acoustic wave is of the form:

\[
p_i = p_0 e^{-j \beta_1 (\cos \theta_1 x + \sin \theta_1 y)},
\]

and

\[
\vec{u}_i = \frac{1}{\omega \rho_1^*} \vec{\beta}_s p_i,
\]

\[
= \frac{\beta_1 \rho_0}{\omega \rho_1^*} (\cos \theta_1 \hat{x} + \sin \theta_1 \hat{y}) e^{-j \beta_1 (\cos \theta_1 x + \sin \theta_1 y)}.\]
Then, the reflected acoustic wave can be written as:

$$p_r = R p_0 e^{-j \beta_1 (\cos \theta_1 x + \sin \theta_1 y)},$$

(18)

and

$$\vec{u}_r = \frac{\beta_1 R p_0}{\omega \rho_1} (-\cos \theta_1 \hat{x} + \sin \theta_1 \hat{y}) e^{-j \beta_1 (\cos \theta_1 x + \sin \theta_1 y)}.$$  

(19)

Whereas the transmitted, within the PML medium, acoustic wave is:

$$p_t = p_{tx} + p_{ty} + p_{tz} = T_x p_0 e^{-j \beta_2 (s_x \cos \theta_2 x + s_y \sin \theta_2 y)} + T_y p_0 e^{-j \beta_2 (s_x \cos \theta_2 x + s_y \sin \theta_2 y)} + 0,$$

(20)

and

$$\vec{u}_t = \frac{1}{\omega \rho'_2} \beta_2 (p_{tx} + p_{ty}) = \beta_2 (T_x + T_y) p_0$$

$$\times (\cos \theta_2 \hat{x} + \sin \theta_2 \hat{y}) e^{-j \beta_2 (s_x \cos \theta_2 x + s_y \sin \theta_2 y)},$$

(21)

with

$$\beta_2^2 = \left(\frac{s_x}{\beta_2}\right)^2 + \left(\frac{s_y}{\beta_2}\right)^2.$$  

(22)

The boundary conditions at an interface of two different media in acoustics impose the continuity of pressure and the normal component of velocity. Therefore, at the interface for \(x = 0\) we obtain:

$$p_i|_{x=0} = p_t|_{x=0},$$

(23)

$$\vec{u}_i + \vec{u}_t \cdot \hat{n}|_{x=0} = \vec{u}_i \cdot \hat{n}|_{x=0},$$

(24)

where \(\hat{n}\) is the normal to the interface unit vector, which coincides with \(\hat{x}\). Aiming to find the conditions for zero reflections from the interface, we either can set the reflected wave to zero then search for these conditions, or initially calculate the reflection coefficient \(R\) in order to set it to zero afterward. Following the former way, assuming that \(R = 0\), in medium 1 there is only the incident wave. Using the above expressions, (23) and (24), can easily give by phase and amplitude matching of the two parts:

$$p_i|_{x=0} = p_t|_{x=0} \Rightarrow \left\{ \begin{array}{l} \beta_1 \sin \theta_1 = \beta_2 s_y \sin \theta_2 \\ T_x + T_y = 1 \end{array} \right.,$$

and

$$u_i,x|_{x=0} = u_t,x|_{x=0} \Rightarrow \frac{\beta_1 \cos \theta_1}{\rho'_1} = \frac{\beta_2 \cos \theta_2}{\rho'_2} (T_x + T_y).$$

Hence:

$$\left\{ \begin{array}{l} \beta_1 \sin \theta_1 = \beta_2 s_y \sin \theta_2 \\ \beta_1 \cos \theta_1 \rho'_2 = \beta_2 \cos \theta_2 \rho'_1 \end{array} \right.$$  

(25)

These should hold such that the interface will be reflectionless for every possible angle \(\theta_1\). We conclude to the same equations calculating first the reflection coefficient and enforcing it to be zero afterward. In this case, after some computational effort, we derive from (23) and (24) the following reflection coefficient:

$$R = \frac{\rho'_2 \beta_1 \cos \theta_1 - \rho'_1 \beta_2 \cos \theta_2}{\rho'_2 \beta_1 \cos \theta_1 + \rho'_1 \beta_2 \cos \theta_2}.$$  

Because now \(\sin^2 \theta_1 + \cos^2 \theta_1 = 1\), solving the above equations for \(\sin \theta_1\) and \(\cos \theta_1\), respectively, we derive:

$$\left(\frac{\beta_2 s_y}{\beta_1}\right)^2 \sin^2 \theta_2 + \left(\frac{\beta_2 \rho'_1}{\beta_1 \rho'_2}\right)^2 \cos^2 \theta_2 = 1,$$

which is valid only if:

$$\left\{ \begin{array}{l} \beta_1 = \beta_2 s_y \\ \beta_1 \rho'_2 = \beta_2 \rho'_1 \end{array} \right.$$  

(26)

Note that no coordinate stretching has been chosen toward the \(y\) and \(z\) directions. So \(s_y = 1, (s_z = 1\) as well) and \(\beta_1 = \beta_2, \rho'_1 = \rho'_2\). Also, we get from (11) that \(\kappa'_1 = \kappa'_2\) and \(\kappa_1 = \kappa_2, \rho_1 = \rho_2, \alpha_1 = \alpha_2, \alpha'_1 = \alpha'_2\).

As a conclusion, there is a substantial degree of freedom regarding the attenuation parameter \(s_x\) and the whole problem reduces to the choice of \(s_x\) such that the wave transmitted in the PML and passed through the normal to the \(x\)-axis interface is further dissipated. The coefficient \(s_x\), which is generally a complex number, can be chosen to be of the form:

$$s_x(x) = s_{0x}(x) \left[ 1 + \frac{\gamma_x(x)}{j \omega \kappa_1} \right],$$

(27)

where \(s_{0x}, \gamma_x\) are functions of \(x\), and they will be selected after the proper numerical experimentation taking into account bibliography as well.

In the case of an interface that is normal to the \(y\)-axis (or \(z\)-axis), \(s_x\) and \(s_z\) (or \(s_y\)) are set to be equal to unity.
The $s_\xi$ in (28) is this of (31), and the $s_\xi$ in (29) is this of (30). However, it is the same $s_\xi$ and the new matching condition is:

$$\frac{\gamma_\xi(\xi)}{\kappa_0} = \frac{\gamma_\xi^*(\xi)}{\rho_0}. \quad (32)$$

So, the parameters $\kappa_0$, $\rho_0$, $\alpha_0$, and $\alpha_0^*$ are arbitrary, and there is no need to satisfy any relation of the form $\alpha_0^*/\rho_0 = \alpha_0/\kappa_0$, which is the matching condition of the original PML responsible for the limitation of the method performing for lossless media only as $\alpha_0^* = 0$. Substituting these in (28) and (29), the PML equations become:

$$\begin{align*}
\frac{\partial (p_x + p_y + p_z)}{\partial \xi} &= - s_{0\xi}(\xi) \left[ j \omega \rho_0 + \alpha_0^* + \gamma_\xi(\xi) \right] u_\xi, \quad (33) \\
\frac{\partial u_\xi}{\partial x} &= - s_{0\xi}(\xi) \left[ j \omega \kappa_0 + \alpha_0 + \gamma_\xi(\xi) \right] p_\xi, \quad (34)
\end{align*}$$

with

$$\frac{\gamma_\xi(\xi)}{\kappa_0} = \frac{\gamma_\xi^*(\xi)}{\rho_0}, \ (\xi = x \text{ or } y \text{ or } z), \quad (35)$$

the matching condition.

Such generalized PML equations are more difficult to be discretized because they involve an additional term of the form $1/j\omega$, which in the time domain implies time integration. The most usual way to tackle with this is to introduce proper auxiliary variables, such as:

$$u^I_\xi = \frac{1}{j\omega} u_\xi, \ p^I_\xi = \frac{1}{j\omega} p_\xi, \ (\xi = x \text{ or } y \text{ or } z).$$

Using these variables, the PML equations in the time domain can be written as shown in (36)–(41) (see next page) accompanied by:

$$\begin{align*}
\frac{\partial u^I_\xi}{\partial t} &= u_\xi, \quad (42) \\
\frac{\partial p^I_\xi}{\partial t} &= p_\xi. \quad (43)
\end{align*}$$

All these equations now can be discretized using a proper FD/TD scheme and updated at every time step, as the method implies. That is, (36) is transformed:

$$u^{n+\frac{1}{2}}_{x,i,j,k} = C_1 u^{n-\frac{1}{2}}_{x,i,j,k} + i \left( C_2 + C_3 \right) \left[ p^{n+\frac{1}{2}}_{x,i,j,k} - p^{n-\frac{1}{2}}_{y,i,j,k} \right]$$

$$+ p^{n+\frac{1}{2}}_{y,i,j,k} \left[ -p^{n-\frac{1}{2}}_{x,i,j,k} - p^{n-\frac{1}{2}}_{y,i,j,k} \right] \quad (44)$$

In a similar way, (39) gives:

$$p^{n+\frac{1}{2}}_{x,i,j,k} = C_1 p^n_{y,i,j,k} - C_2 p^{n+\frac{1}{2}}_{x,i,j,k} + C_3 \left[ u^{n+\frac{1}{2}}_{x,i,j,k} - u^{n-\frac{1}{2}}_{x,i,j,k} \right]. \quad (45)$$
where $\xi = x$. The PML absorbs the acoustic wave in a 2-D lossless medium. The PML will be deduced, its ability in the absorption of acoustic waves and the minimization of reflections caused by the truncation boundaries remains the same irrespective of the kind of medium (lossless or lossy) that concerns us.

To start with, let us examine the propagation of an acoustic wave in a 2-D lossless medium. The PML absorbers of the original form are constructed ($s_x = s_y = 1$) to enclose the area of propagation, which is divided to an aggregate of $143 \times 143$ sample points $[x\text{-axis: 1}\ldots N, \ y\text{-axis: 1}\ldots N, \ N = 143]$. At the center of the FDTD grid [i.e., point $(72,72)$], a point source is considered. It is a Gaussian pulse excitation of the form: $p(x,y,t) = \text{I}_0 \cdot \exp\left[-\left(\frac{x^2}{\alpha^2} + \frac{y^2}{\alpha^2}\right)\right]$...N

\[
\begin{align*}
\frac{\partial (p_x + p_y + p_z)}{\partial x} &= -s_{0x}(x) \left\{ \rho_0 \frac{\partial u_x}{\partial t} + \left[ \alpha_0^* + \gamma_x^*(x) \right] u_x + \frac{\alpha_0^* \gamma_x^*(x)}{\rho_0} \right\}, \\
\frac{\partial (p_x + p_y + p_z)}{\partial y} &= -s_{0y}(y) \left\{ \rho_0 \frac{\partial u_y}{\partial t} + \left[ \alpha_0^* + \gamma_y^*(y) \right] u_y + \frac{\alpha_0^* \gamma_y^*(y)}{\rho_0} \right\}, \\
\frac{\partial (p_x + p_y + p_z)}{\partial z} &= -s_{0z}(z) \left\{ \rho_0 \frac{\partial u_z}{\partial t} + \left[ \alpha_0^* + \gamma_z^*(z) \right] u_z + \frac{\alpha_0^* \gamma_z^*(z)}{\rho_0} \right\},
\end{align*}
\]

and (42) for $\xi = x$ becomes:

\[
u_{x,1}^n \left(i + \frac{1}{2}, j, k\right) = u_{x,1}^{n-1} \left(i + \frac{1}{2}, j, k\right) + \Delta t u_x^{n-\frac{1}{2}} \left(i + \frac{1}{2}, j, k\right),
\]

where

\[
\begin{align*}
C_1^* &= \frac{\rho_0}{\Delta t} - \frac{\alpha_0^* + \gamma_x^*}{2}, \\
C_2^* &= \frac{\rho_0}{\Delta t} + \frac{\alpha_0^* + \gamma_x^*}{2}, \\
C_3^* &= \frac{1}{s_{0x} \Delta x \left( \frac{\rho_0}{\Delta t} + \frac{\alpha_0^* + \gamma_x^*}{2} \right)}, \\
C_1 &= \frac{\kappa_0}{\Delta t} - \frac{\alpha_0 + \gamma_x}{2}, \\
C_2 &= \frac{\kappa_0}{\Delta t} + \frac{\alpha_0 + \gamma_x}{2}, \\
C_3 &= \frac{1}{s_{0x} \Delta x \left( \frac{\kappa_0}{\Delta t} + \frac{\alpha_0 + \gamma_x}{2} \right)}.
\end{align*}
\]

V. SIMULATION RESULTS

The PML developed was validated in many problems of acoustic propagation and scattering in 2-D and 3-D. As will be deduced, its ability in the absorption of acoustic waves and the minimization of reflections caused by the truncation boundaries remains the same irrespective of the kind of medium (lossless or lossy) that concerns us.

To start with, let us examine the propagation of an acoustic wave in a 2-D lossless medium. The PML absorbers consist of $143$ sample points ($x$-axis: 1, ..., $N$, $y$-axis: 1, ..., $N$, $N = 143$). At the center of the FDTD grid [i.e., point $(72,72)$], a point source is considered. It is a Gaussian pulse excitation of the form: $s(x,y) = c - \frac{\alpha_0}{\Delta t} \frac{\alpha_0 + \gamma_x}{2}$, in which $\alpha$ is the attenuation coefficient, and $n_0$, $\text{decay}$ constant quantities. The parameters and their values for the simulation problem are reported in Table I. The PML medium consists of $m$ layers, and the parameter of losses is chosen to be of parabolic variation along these layers in order to accomplish the best absorption. Therefore, from the inner to the outer layer, the value of $\alpha$ increases from zero to $\alpha_{\text{max}}$ according to the form:

\[
\alpha(i) = \alpha_{\text{max}} \cdot \left( \frac{m + 1 - i}{m} \right)^2, \quad i = 1,\ldots,m.
\]

Fig. 3 shows the variation of the attenuation parameter $\alpha$ from layer to layer in the PML regions. The maximum value of the attenuation coefficient $\alpha_{\text{max}}$ has been selected after some computational effort estimating the global error. It was found that this value must not be either too big or too small. If it is big enough, there will be numerical reflections because of the sharp variation from zero to $\alpha_{\text{max}}$. But, if it is quite small, the PML layers cannot manage to preserve satisfactory absorption of the transmitted waves; therefore, more layers are needed to achieve the desired absorption. Fig. 4(a) shows how the global error of the FDTD method is reduced by the decrease of the attenuation value $\alpha_{\text{max}}$. On the contrary, Fig. 4(b) shows

\[
\begin{array}{|c|c|c|}
\hline
\text{Parameter} & \text{Value} & \text{Measurement units} \\
\hline
\rho & 1500 & \text{m sec}^{-1} \\
\rho & 1000 & \text{Kg m}^{-3} \\
\kappa & 4.211 \cdot 10^{-10} & \text{m}^{2} \text{N}^{-1} \\
f & 1 & \text{MHz} \\
\alpha_{\text{max}} & 3.212 \cdot 10^{-3} & \text{m}^{2} \text{N}^{-1} \text{sec}^{-1} \\
\lambda & c/f, \Delta h = \lambda/10, \Delta t = \Delta h/(2c) \\
\hline
\end{array}
\]
the increase of the global error by the constant reduction of the attenuation value.

The global error is estimated at each time step by the double summation:

$$\sum_{i=m+1}^{N-m} \sum_{j=m+1}^{N-m} [TRPS(i, j) - PS(i, j)]^2 / (N - m)^2,$$

where, $TRPS(i, j)$ and $PS(i, j)$ are the true and the experimental value of the acoustic pressure at the $(i, j)$ point of the FDTD grid, respectively. The true values of the acoustic pressure are computed at each point of a much larger FDTD grid of $1067 \times 1067$ sample points. The corresponding moment that the experimental values are computed, the acoustic waves in such a grid have not yet reached the truncation boundaries, and the reflections have not been produced in order to corrupt the propagated waves. Thus, the real values of the fields can be computed and compared with those of the experiment showing the successful simulation of the unbounded space by the use of the PML as an ABC.

What is more, the great ability of the PML technique is the possibility for further reduction of the reflections, by increasing the number of the PML layers in the FDTD grid. The effects of this variation on global error are shown in Fig. 5 for different number of PML layers.

The visualization of the acoustic field, and particularly of the acoustic pressure values at each point of the FDTD grid, are shown in Fig. 6 for different time steps. The field before striking at the boundaries and after this is presented. It is clear that the propagation of the acoustic waves continues like the truncation—termination—of the FDTD space does not exist.

To show that the transition from 2-D to 3-D is straightforward, the same problem was solved for a 3D ($143 \times 143 \times 143$ cells) FDTD domain. In Fig. 7 it can be seen clearly that analogous results have been taken for this case also. These depictions concern the acoustic field at different levels of constant $z$ and at the same time step. In Fig. 7(c), the acoustic pressure is illustrated for the central $z$ level of the cubic FDTD domain, in the center of which the point source has been located; therefore, the acoustic waves propagate further than these of the upper levels [Figs. 7(a) and (b)].

Nevertheless, this form of PML does not work satisfyingly when the propagation medium is lossy, which usually occurs in acoustics. In Fig. 8(a), it is apparent that significant reflections can be caused at the truncation boundaries when losses are considered. To overcome this drawback, the stretched coordinate system is assumed ($s_x, s_y, s_z \neq 1$), and the functions $s_0 \xi$, $\gamma_\xi$, in (34), are set to be:

$$s_0 \xi(\xi) = 1 \text{ and } \gamma_\xi(i, j, k) = \alpha_{\text{max}} \left( \frac{m + 1 - \delta}{m} \right)^2,$$
Fig. 5. Global error for different number of PML layers. There is a significant reduction of the global error, increasing the number of the layers.

Fig. 6. The acoustic pressure field at different time steps for the 2-D case, (a) $n = 130$, (b) $n = 180$.

Fig. 7. The acoustic pressure at different $z$ levels for the 3-D case. (a) $z = 110$, (b) $z = 90$, (c) $z = 72$. 
where $\xi = x$ or $y$ or $z$ and $\delta = i$ or $j$ or $k$, respectively, taking values between 1 and $m$.

The acoustic wave propagation in a lossy medium with $\alpha_0 = 3.212 \cdot 10^{-5}$ and $\alpha_0^* = 0$ is simulated for the same FDTD grid, the same number of PML layers with maximum attenuation coefficient $\alpha_{\text{max}} = 3.212 \cdot 10^{-3}$, and the same excitation. Fig. 8(b) shows how the modified PML worked in this case as the reflections observed in Fig. 8(a) no more exist and the acoustic field has not been distorted.

The efficiency of the generalized PML absorbing boundary is further studied in acoustic scattering from an interface of two different lossy media and an obstacle of square cross section existing into the medium of propagation for the 2-D case. The interface is placed at the 50th column of the FDTD grid (points $(50, j)$), and the $20 \times 20$ cells scatterer is placed on the left side of a same FDTD grid (see also Fig. 9). The second lossy medium, which has been considered in these simulations, is characterized by the constants presented in Table II. The PML media, which are going to enclose the second mess, are modified to become suitable for this one and achieve the desired absorption. This means that the construction of the PML does not remain constant for all acoustic meshes, and the parameters have to adjust to each of them.

The scattering acoustic field is presented for both cases in Fig. 9, which indicates high absorption of the scattering.
waves and verifies the usefulness of such a technique as an absorbing boundary.

VI. CONCLUSIONS

In this paper, the generalized form of the PML was developed for 3-D problems of acoustic scattering in order to perform as an ideal ABC in both lossless and lossy media. It was verified that, after proper selection of the parameters, theoretically no reflections of the propagating and scattering waves occur at the truncation boundaries independently of the angle of incidence. The idea of stretched coordinates was used in order to enable the technique to be properly built to act as a free space simulator of lossy media as well. At the same time, the needful FDTD simulation formula was fully developed and presented in detail for the 3-D case.

Several simulations demonstrated the effectiveness of the method in acoustics. But, in most cases, the losses must be taken into account and the original PML would be inadequate to act as an efficient absorbing boundary. Moreover, the great potentiality of the method is the possibility for better tissue characterization and modeling, and thus for better performance at simulations of acoustic scattering and ultrasound imaging. One further aspect in this direction is the successful consideration of losses’ frequency dependence in acoustic scattering simulations of dispersive media, which is currently under investigation.

REFERENCES


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