Analytical and Numerical Solution of the Eddy-Current Problem in Spherical Coordinates Based on the Second-Order Vector Potential Formulation

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Abstract—The three-dimensional (3-D) eddy-current problem, described in spherical coordinates, is studied both analytically and numerically. Since the vector field equation is not separable in the spherical coordinate system, the second-order vector potential (SOVP) formulation is used to treat the problem by reducing it to the solution of the scalar field equation. While the analytical solution is expressed in terms of known orthogonal expansions, the numerical solution utilizes the finite difference method. Examples of engineering applications are provided, concerning computation of eddy-current distribution in a conducting sphere by a filamentary excitation of arbitrary shape.

Index Terms—Analytical, eddy currents, finite-difference method.

I. INTRODUCTION

The question of the separability of vector Helmholtz and Laplace equations in curvilinear coordinate systems is thoroughly discussed in [1] where the proposed method is to split the solution in two components: the longitudinal and the transverse. This method is quite advantageous not only because it provides separated solutions, but simplifies the imposition of boundary conditions as well.

Following this procedure, Smythe [2] and Hammond [3] expressed the magnetic vector potential \( \mathbf{A} \) as the curl of a new second-order vector potential \( \mathbf{W} \). This potential can be resolved in two components normal to each other which satisfy the scalar Helmholtz or Laplace equations. Thus, the problem of the solution of the vector equation is reduced to the solution of the respective scalar one which is separable in a number of coordinate systems.

Previous work, concerning analytical solutions for eddy currents implementing the second-order vector potential (SOVP) formulation, has been performed for problems described in other coordinate systems. In [4] and [5], the SOVP formulation for the Cartesian coordinate system is presented to treat the problem of eddy-current evaluation in an infinite conducting halfspace by a current source of arbitrary shape, while in [6] and [7], exact analytical solutions, utilizing the above formulation, are presented for the same problem and for the case of a circular source normal to the halfspace.

In [8] and [9], the SOVP formulation for the cylindrical coordinate system is presented to study the problem of eddy-current evaluation in an infinite conducting cylinder. In [10], exact analytical expressions are presented for the case of a saddle-shaped source and in [11] for an external electromagnetic field normal to the cylinder’s surface.

In the spherical coordinate system, the SOVP formulation for the evaluation of eddy currents has not been implemented yet. Analytical solutions for the eddy currents induced in a conducting sphere are presented for the two-dimensional case of a circular source around a sphere in [12], and the same problem is solved for a sphere whose conductivity varies continuously with the radial distance in [13]. In [14], an alternative method based on the conventional vector potential is used to solve the three-dimensional (3-D) problem of eddy-current evaluation in a conducting sphere by a source of arbitrary shape having radial current elements.

In the present paper, the separability advantages of the SOVP formulation are utilized to provide solutions for the 3-D eddy-current problem described by the vector Helmholtz and Laplace equations in the spherical coordinate system for processes which remain harmonic with time. More specifically, analytical expressions are derived for the eddy currents induced in a conducting sphere by filamentary excitations of particular shape.

In the case of excitations of arbitrary shape when the problem cannot be solved analytically, a numerical method is used based on finite-difference method (FDM). Numerical methods, utilizing the SOVP formulation, have received little attention. The finite-element method (FEM) is proposed in [15], In [16]–[20], the boundary-element method (BEM) is used, while in [21], the eddy-current problem is treated by the FDM.

In the following paragraphs, after a brief presentation of the implementation of the SOVP formulation in spherical coordinates, the analytical and numerical treatments of the problem studied are presented. Eventually, it is shown that only one scalar quantity is needed for the description of the electric field and, therefore, the eddy currents throughout the conducting sphere. This is a very important conclusion that simplifies the problem considerably.

II. THE SOVP FORMULATION

The electromagnetic field quantities in 3-D steady-state problems can be derived from the solution of the vector
Helmholtz equation for the magnetic vector potential \( \mathbf{A} \)
\[
\nabla^2 \mathbf{A} + k^2 \mathbf{A} = 0
\]  
(1)
where \( k^2 = -j \omega \mu \sigma \) in the quasistatic limit. The drawback with the solution of the vector equation (1), when using spherical or generally curvilinear coordinates, is the coupling between the components of \( \mathbf{A} \) in the three resulting scalar equations. This inconvenience is avoided by using the second-order vector potential \( \mathbf{W} \) which for a solenoidal \( \mathbf{A} \) in spherical coordinates is defined as
\[
\mathbf{A} = \nabla \times \mathbf{W} = \nabla \times [(r W_a) \mathbf{r}_a + \mathbf{r}_a \times \nabla (r W_b)]
\]  
(2)
where \( W_a \) and \( W_b \) are scalar functions that satisfy the scalar Helmholtz equations
\[
\nabla^2 W_a + k^2 W_a = 0
\]  
(3)
\[
\nabla^2 W_b + k^2 W_b = 0.
\]  
(4)
Since the above two equations are separable in a number of coordinate systems, including the spherical one, the formulation based on the SOVP can be used for the separation of the vector Helmholtz equation.

Equation (2) results in the following expressions for the components of \( \mathbf{A} \)
\[
A_r = \cot \theta \frac{1}{r} \frac{\partial W_b}{\partial r} + \frac{1}{r} \frac{\partial^2 W_b}{\partial \theta^2} + \frac{1}{r \sin^2 \theta} \frac{\partial^2 W_b}{\partial \phi^2}
\]  
(5)
\[
A_\theta = -\frac{1}{r \sin \theta} \frac{\partial W_a}{\partial \phi} - \frac{1}{r} \frac{\partial^2 W_b}{\partial \theta \partial r} - \frac{1}{r} \frac{\partial W_b}{\partial \theta}
\]  
(6)
\[
A_\phi = -\frac{1}{r \sin \theta} \left( \frac{\partial^2 W_b}{\partial \phi \partial r} + \frac{1}{r} \frac{\partial W_b}{\partial \phi} \right)
\]  
(7)
The magnetic flux density can be expressed in terms of \( W_a \) and \( W_b \) in spherical coordinates as
\[
\mathbf{B} = \nabla \times \mathbf{A} = \nabla \times \nabla \times \mathbf{W}
\]  
(8)
which results in the following expressions for the components of \( \mathbf{B} \)
\[
B_r = \frac{1}{r} \frac{\partial^2 W_a}{\partial \theta^2} + \frac{2}{r} \frac{\partial W_a}{\partial r} + k^2 W_a r
\]  
(9)
\[
B_\theta = \frac{1}{r} \frac{\partial W_a}{\partial r} + \frac{1}{r} \frac{\partial W_a}{\partial \theta} - \frac{k^2}{r} \frac{\partial W_b}{\partial \theta}
\]  
(10)
\[
B_\phi = \frac{1}{r \sin \theta} \left( \frac{\partial^2 W_a}{\partial \phi \partial r} + \frac{1}{r} \frac{\partial W_a}{\partial \phi} \right) + k^2 \frac{\partial W_b}{\partial \phi}.
\]  
(11)
From the above equations, it is observed that, for a nonconducting region (\( k = 0 \)), the magnetic flux density depends only upon \( W_a \) and is expressed as the gradient of a scalar quantity
\[
\mathbf{B} = \nabla \left[ \frac{\partial (r W_a)}{\partial r} \right],
\]  
(12)
Fig. 1. The geometry under consideration. A filamentary excitation of arbitrary shape near a solid conducting sphere.

The electric field intensity and, thus, the eddy currents are derived from
\[
\mathbf{J} = \sigma \mathbf{E} = -j \omega \sigma \mathbf{A} = -j \omega \sigma \nabla \times \mathbf{W}.
\]  
(13)
The above analysis indicates the inherent advantages of the SOVP formulation. Indeed, when solving the problem analytically, the solution of the vector equations can be derived from the solution of the respective scalar ones which are separable in a number of coordinate systems.

When solving the problem numerically, other formulations suffer to a great extent from the fact that 3-D eddy-current problems are represented in terms of vector equations which dramatically increase the number of unknowns and consequently the computer storage and central processing unit (CPU) time requirements. In the SOVP formulation, two scalars are needed for the conducting region, both satisfying the scalar Helmholtz equation and one scalar for the nonconducting region, satisfying the scalar Laplace equation. Therefore, the advantages of this approach are: a) only Laplacian operators are involved, providing thus the minimum computational cost, and b) the least possible number of degrees of freedom per node is required.

III. SETUP OF THE PROBLEM

The geometry of the problem studied is shown in Fig. 1. A filamentary excitation of arbitrary shape, driven by a harmonically varying source current \( I \exp(j \omega t) \), is located near a nonmagnetic conducting sphere of radius \( b \). The conductivity \( \sigma \) of the sphere is constant throughout its volume.

The magnetic vector potential depends upon \( W_a \) and \( W_b \) in both conducting and nonconducting regions as observed from (5), (6), and (7), while the magnetic flux density depends on \( W_a \) and \( W_b \) in the conducting and only on \( W_a \) in the nonconducting regions.

Physically realistic solutions must be consistent with the usual boundary conditions across the interface \( r = b \), namely, the continuity of the tangential components of the magnetic
field intensity and of the normal component of the magnetic flux density. We find that they do hold if the following conditions for the potentials \( W_{a} \) and \( W_{b} \) also hold at the interface \( r = b \) between the conducting-2 and the nonconducting-1 regions

\[
\begin{align*}
W_{a,1} &= W_{a,2}, \quad r = b \\
\frac{\partial W_{a,1}}{\partial r} &= \frac{\partial W_{a,2}}{\partial r}, \quad r = b \\
\frac{\partial W_{b,2}}{\partial r} &= 0, \quad r = b \\
W_{b,2} &= 0, \quad r = b.
\end{align*}
\]

As \( W_{b,2} \) and its normal gradient vanish at the interface and there are no sources in the conducting region, \( W_{b,2} \) is zero throughout the sphere. Hence, the magnetic vector potential in the conductor is simply

\[
A_2 = \nabla \times W_2 = \nabla \times [r W_{a,2} \mathbf{r}_0]
\]

which combined with (13) implies that the induced eddy currents do not have a radial component throughout the sphere and, therefore, they flow parallel with its surface.

The potential in the air, \( W_{a,1} \), can be expressed as the sum of the primary potential \( W_{a,s} \) due to the source current in free space and the secondary potential \( W_{a,ec} \) originated from the eddy currents induced within the conducting sphere

\[
W_{a,1} = W_{a,ec} + W_{a,s}.
\]

The standard procedure for the calculation of \( W_{a} \) in the two regions is to express it as series of known orthogonal expansions in the two regions. The unknown expansion coefficients are determined from the following equations, if we replace \( \mathbf{W}_{a,1} \) and \( \mathbf{W}_{a,2} \) in (14) and (15) by its expression in (19)

\[
\begin{align*}
W_{a,ec} + W_{a,s} &= W_{a,2}, \quad r = b \\
\frac{\partial (W_{a,ec} + W_{a,s})}{\partial r} &= \frac{\partial W_{a,2}}{\partial r}, \quad r = b.
\end{align*}
\]

Once the potential \( W_{a,2} \) is determined, any electromagnetic field quantity inside the conducting sphere can be calculated. In the problem studied, the induced eddy currents can be calculated from (13) and (18).

### IV. Analytical Treatment

The potential produced in the nonconducting region by the eddy currents themselves, \( W_{a,ec} \), is the solution of the scalar Laplace equation in the spherical coordinate system, which results from (3) for \( k = 0 \). Since this region is defined by \( r \geq b \), the following expression is used [22] as it ensures that the potential vanishes at infinite radial distances

\[
W_{a,ec} = \sum_{n=0}^{\infty} \sum_{m=0}^{n} \frac{1}{n+1} P_n^{m} \left( \cos \theta \right) \left[ S_{ec} \sin m\phi + C_{ec} \cos m\phi \right].
\]

The potential inside the conducting sphere, \( W_{a,2} \), is the solution of the scalar Helmholtz equation (3) in the spherical coordinate system. Since this region, defined by \( r \leq b \), involves the center of the sphere, the following expression is used [22]:

\[
W_{a,2} = \sum_{n=0}^{\infty} \sum_{m=0}^{n} i_n(kr) P_n^{m} \left( \cos \theta \right) S_{2} \sin m\phi + C_{2} \cos m\phi.
\]

in order to ensure a finite value for the potential at zero radial distance. In the above equations, \( i_n(kr) \) is the modified spherical Bessel function of the first kind, which remains finite for \( r = 0 \), and \( P_n^{m} \left( \cos \theta \right) \) is the associated Legendre function of the first kind [23].

To evaluate the SOVP, the unknown expansion coefficients \( S_{ec}, C_{ec}, S_{2}, \) and \( C_{2} \) have to be determined. This is achieved by imposing the boundary conditions (20) and (21) on the dividing surface \( r = b \). Thus, it is necessary to find an expression for the potential produced by the source in free space \( W_{a,s} \) similar to (22) and (23). For the moment, the potential in the free space region defined by \( r \leq b \) is assumed to be of the following form

\[
W_{a,s} = \sum_{n=0}^{\infty} \sum_{m=0}^{n} r^n P_n^{m} \left( \cos \theta \right) S_{s} \sin m\phi + C_{s} \cos m\phi
\]

where \( S_{s} \) and \( C_{s} \) are known expansion coefficients that will be computed later. The final expressions for the unknown expansion coefficients are

\[
\begin{align*}
\begin{bmatrix} S_2 \\ C_2 \end{bmatrix} &= \begin{bmatrix} S_s \\ C_s \end{bmatrix} \frac{(2n+1)\hat{n}^n}{i_{n-1}(k\hat{b})} \\
\begin{bmatrix} S_{ec} \\ C_{ec} \end{bmatrix} &= \begin{bmatrix} S_s \\ C_s \end{bmatrix} \frac{i_{n+1}(k\hat{b})}{i_{n-1}(k\hat{b})}.
\end{align*}
\]

Up to this point, we have determined the potential in all regions as a function of the expansion coefficients \( S_{s} \) and \( C_{s} \), shown in (24), which expresses the potential in free space from a source of arbitrary shape. The procedure for the analytical calculation of this potential for a number of sources with a special shape is presented in the following paragraph.

Once the potential \( W_{a,2} \) has been calculated, the induced eddy currents can be deduced by combining (13) and (18).

The final expressions for the latitudinal \((\theta)\) and longitudinal \((\phi)\) components of the eddy-current density are

\[
\begin{align*}
J_\theta &= \frac{j \omega S_{ec} \sin \theta}{\sin \theta} \sum_{n=0}^{\infty} \sum_{m=0}^{n} \frac{i_n(kr)(2n+1)\hat{n}^n}{i_{n-1}(k\hat{b})} P_n^{m} \left( \cos \theta \right) \left[ S_{ec} \sin m\phi + C_{ec} \cos m\phi \right] \\
J_\phi &= \frac{j \omega S_{ec} \sin \theta}{\sin \theta} \sum_{n=0}^{\infty} \sum_{m=0}^{n} \frac{i_n(kr)(2n+1)\hat{n}^n}{i_{n-1}(k\hat{b})} \frac{dP_n^{m} \left( \cos \theta \right)}{d\theta} \left[ S_{ec} \sin m\phi + C_{ec} \cos m\phi \right].
\end{align*}
\]

In (28), as well as in all following equations, the derivative of the associated Legendre function of the first kind can be calculated from Legendre functions themselves, by

\[
\frac{dP_n^{m} \left( \cos \theta \right)}{d\theta} = \frac{n + m}{\sin \theta} P_n^{m} \left( \cos \theta \right) - \frac{n + m}{\sin \theta} P_{n-1}^{m} \left( \cos \theta \right).
\]
V. THE POTENTIAL IN FREE SPACE

In the preceding paragraph, an expression in the form of double series was necessary for the potential \( W_{o,s} \) produced in free space at \( r = b \) by an arbitrary shaped source. In this paragraph, such an expression will be derived for the region with a radial distance smaller than the one of all source points as it includes the dividing boundary surface. The whole procedure begins from the Biot–Savart law for the magnetic flux density which is written as

\[
B = \frac{\mu_0 I}{4\pi} \nabla \times \int_C \frac{d\ell}{R} \tag{30}
\]

where the integration is performed over the path followed by the excitation current and \( R \) denotes the distance between a point on the excitation \((r_o, \theta_o, \phi_o)\) and the point \((r, \theta, \phi)\) where the calculation of the field is required

\[
R^2 = r^2 + r_o^2 - 2rr_o \left[ \cos \theta \cos \theta_o + \sin \theta \sin \theta_o \cos (\phi - \phi_o) \right] \tag{31}
\]

The use of the Stokes theorem transforms the line integral of (30) to a surface integral

\[
B = \frac{\mu_0 I}{4\pi} \nabla \times \oint_S d\mathbf{s} \times \nabla_o \left( \frac{1}{R} \right) \tag{32}
\]

where the gradient operator \( \nabla_o \) acts with respect to the source coordinates. In the above equations, \( d\mathbf{l} \) is the differential distance vector tangential to the path of the source current \( C \), and \( d\mathbf{s} \) is the differential area vector normal to any surface bounded by the path \( C \).

After some algebra, (32) takes the following form which expresses the magnetic flux density as the gradient of a scalar quantity involving a surface integral

\[
B = \nabla \left[ -\frac{\mu_0 I}{4\pi} \oint_S d\mathbf{s} \cdot \nabla_o \left( \frac{1}{R} \right) \right] \tag{33}
\]

If (33) is compared to (12) for the expression of the magnetic flux density in a nonconducting region, the potential in free space becomes

\[
W_{o,s} = -\frac{\mu_0 I}{4\pi r} \oint_S d\mathbf{s} \cdot \nabla_o \left( \int \frac{d\mathbf{r}}{R} \right). \tag{34}
\]

If additionally \( 1/R \) is replaced with its well-known equivalent in spherical coordinates

\[
\frac{1}{R} = \sum_{n=0}^{\infty} \sum_{m=0}^{n} \varepsilon_m \frac{(n-m)!}{(n+m)!} P_n^m(\cos \theta)P_n^m(\cos \theta_o) \cdot \cos[m(\phi - \phi_o)] \frac{r^n}{r_o^{n+1}}, \quad r \leq r_o
\]

\[
\varepsilon_m = \begin{cases} 1, & m = 0 \\ 2, & m \neq 0 \end{cases}
\]

the potential can be calculated from the analytical surface integration, as shown in (34). In this equation the differential area vector \( d\mathbf{s} \) can be replaced by its expression in the spherical system of coordinates

\[
d\mathbf{s} = \frac{r_o^2 \sin \theta_o}{r^2} d\theta_o d\phi_o + r_o \sin \theta_o d\phi_o d\theta_o + r_o d\theta_o d\theta_o \cos \phi_o.
\]

Results in the form of double series expansions can be derived if the surface of integration and, therefore, of the excitation lies on one of the three surfaces of the spherical coordinate system.

1) The first excitation examined is located on a plane of constant radius \( r_o \) extending from \( \theta_1 \) to \( \theta_2 \) in the latitudinal and from \( \phi_1 \) to \( \phi_2 \) in the longitudinal direction. For this excitation, named here \( r_o \)-excitation and shown in Fig. 2, the integration of (34) gives the following result for the potential

\[
W_{o,s} = \frac{\mu_0 I}{4\pi r_o} \sum_{n=0}^{\infty} \sum_{m=0}^{n} \varepsilon_m \frac{(n-m)!}{(n+m)!} P_n^m(\cos \theta)P_n^m(\cos \theta_o) \cdot \frac{r^n}{mr_o^{n+1}} \left[ (\sin m\phi - \sin m\phi_o) \cos m\phi - (\cos m\phi - \cos m\phi_o) \sin m\phi \right] \tag{37}
\]

where

\[
P_n^m(\cos \theta_o) = \int_{\theta_o}^{\theta_2} P_n^m(\cos \theta_o) \sin \theta_o d\theta_o.
\]

2) The second excitation examined is located on a plane of constant latitude \( \theta_o \) extending from \( r_2 \) to \( r_2 \) in the radial and from \( \phi_1 \) to \( \phi_2 \) in the longitudinal direction. For this excitation, named here \( \theta_o \)-excitation and shown in Fig. 3, the integration of (34) gives the following result for the potential

\[
W_{o,s} = \frac{\mu_0 I}{4\pi r_o} \sin \theta_o \sum_{n=0}^{\infty} \sum_{m=0}^{n} \varepsilon_m \frac{(n-m)!}{(n+m)!} P_n^m(\cos \theta_o) \cdot \frac{r^n}{mr_o^{n+1}} \left[ (\sin m\phi - \sin m\phi_o) \cos m\phi - (\cos m\phi - \cos m\phi_o) \sin m\phi \right]. \tag{39}
\]
3) The third excitation examined is located on a plane of constant longitude \( \phi_0 \) extending from \( \varphi_1 \) to \( \varphi_2 \) in the radial and from \( \theta_1 \) to \( \theta_2 \) in the latitudinal direction. For this excitation, named here \( \phi_0 \)-excitation and shown in Fig. 4, the integration of (34) gives the following result for the potential

\[
W_{\phi_0,s} = \frac{\mu_0 I}{4\pi} \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \varepsilon_m (n-m)! \frac{P_n^m(\cos \theta)}{(n+m)!} P_D(\theta_1, \theta_2) \cos m\phi_0 \sin m\phi_0 
\]

where

\[
P_D(\theta_1, \theta_2) = \int_{\theta_1}^{\theta_2} \frac{P_n^m(\cos \theta)}{\sin \theta} d\theta.
\]

In all of the above expressions of the potential \( W_{\phi_0,s} \) and for those terms that both numerator and denominator become zero for \( n = 0 \) or \( m = 0 \), the L’Hospital rule is used to avoid the indeterminacy.

Equations (37), (39), and (40) provide the SOVP in free space for three particular excitation shapes. Inspection and comparison of these equations with (24) reveals the coefficients \( C_8 \) and \( S_8 \) needed for the calculation of the SOVP in the sphere and, thus, of the eddy currents induced therein.

VI. Numerical Treatment

In the case of an arbitrary-shaped source current, when the surface integration in (34) cannot be performed analytically for the double series representation of \( W_{\phi_0,s} \), a numerical treatment of the problem based on the SOVP formulation is used.

The FDM [24] is chosen, due to its simplicity for the discretization of the scalar Helmholtz and Laplace equations for \( W_{a_2} \) and \( W_{a,cc} \), respectively. In this numerical treatment, the simpler form of (31) is chosen instead of (35) for the representation of the distance \( R \). After some processing, (34) results in the following expression for the SOVP from a filamentary excitation of arbitrary shape in free space

\[
W_{\phi_0,s} = \frac{\mu_0 I}{4\pi r} \int_S ds \cdot \nabla_0 \ln \left\{ 2R + 2r - 2r_0 \right\} \cos \theta_0 + \sin \theta_0 \sin \theta_0 \cos (\phi - \phi_0) \}
\]

which after the performance of the gradient operator, with respect to the source coordinates, takes the following form shown in (43) at the bottom of the page where

\[
C_1 = \cos \theta_0 \cos \theta_2 + \sin \theta_0 \sin \theta_2 \cos (\phi - \phi_0) \}
\]

\[
C_2 = -\cos \theta_2 \sin \theta_0 + \sin \theta_2 \cos \theta_0 \cos (\phi - \phi_0) \}
\]

\[
C_3 = \sin \theta_0 \sin \theta_2 \sin (\phi - \phi_0) \}
\]

Expression (43) is preferred for the representation of \( W_{\phi_0,s} \) due to its simplicity and reduction of the computation time as it does not involve a double series of higher mathematical functions. The implementation of the FDM is also quite simple and straightforward as the fields depend only on a single scalar quantity. Thus, there is only one unknown quantity in the conducting and nonconducting regions, which is evaluated from the discretization of (3), and there are two unknown quantities on the boundary which are evaluated from the system resulting from the discretization of (19) and (20). Simple central differences and a successive over-relaxation (SOR) scheme are used for the iterative solution of the resulting system.
Fig. 5. Amplitude contours of eddy-current density on the surface of the sphere, (a) $\theta$ component, (b) $\phi$ component, in [A/m$^2$], presented under a Mercator’s projection. Excitation is of the $r_o$-type and the calculations have been performed analytically.
Fig. 6. Amplitude contours of eddy-current density on the surface of the sphere, (a) $\theta$ component, (b) $\phi$ component, in [A/m$^2$], presented under a Mercator’s projection. Excitation is of the $\theta_e$-type and the calculations have been performed analytically.
Fig. 7. Amplitude contours of eddy-current density on the surface of the sphere, (a) $\theta$ component, (b) $\phi$ component, in [A/m$^2$], presented under a Mercator's projection. Excitation is of the $\phi_o$-type, and the calculations have been performed analytically.
VII. NUMERICAL RESULTS

In the analytical treatment, eddy currents are computed for the \( r_0 \)-, \( \theta_0 \)-, and \( \phi_0 \)-excitations respectively. The double series do converge very rapidly, and the whole procedure is straightforward and easy to perform. A public domain package in the form of a FORTRAN library is used to compute the Bessel functions of real order and complex argument [25]. Numerical results for these three excitations are presented for the amplitude of the latitudinal (\( \theta \)) and longitudinal (\( \phi \)) components of the eddy currents on the surface of the sphere. Among the possible projections of the aforementioned surface, Mercator’s projection is used because it is conformal and has the property of preserving angles and, therefore, shapes. This is achieved by projecting the spherical surface \((\theta, \phi)\) on a plane \((x, y)\) under the transformation

\[
x = \phi, \quad y = \ln(\csc \theta + \cot \theta).
\]

The disadvantage of Mercator’s projection is the lack of accuracy near the poles of the sphere which are mapped to infinite distances.

In all cases of the problem studied, the sphere has a radius of \( b = 0.1 \) m and a conductivity equal to that of aluminum, \( \sigma = 3.54 \times 10^7 \) S/m. The excitation current has an amplitude of 1 A and a frequency of 50 Hz.

Fig. 5(a) and (b) show eddy currents induced by an \( r_0 \)-excitation, depicted in Fig. 2, with \( r_0 = 0.12 \) m and which extends a total angle of 90° in the longitudinal direction from \( \theta_1 = 45° \) to \( \theta_2 = 135° \) and a total angle also of 90° in the latitudinal direction from \( \phi_1 = 45° \) to \( \phi_2 = 45° \).

Fig. 6(a) and (b) show eddy currents induced by a \( \theta_0 \)-excitation, depicted in Fig. 3, with \( \theta_0 = 90° \) and which extends from \( r_1 = 0.12 \) m to \( r_2 = 0.15 \) m in the radial direction and a total angle of 90° in the longitudinal direction from \( \phi_1 = 45° \) to \( \phi_2 = 45° \).

Fig. 7(a) and (b) show eddy currents induced by a \( \phi_0 \)-excitation, depicted in Fig. 4, with \( \phi_0 = 90° \) and which extends from \( r_1 = 0.12 \) m to \( r_2 = 0.15 \) m in the radial direction and a total angle of 90° in the latitudinal direction from \( \theta_1 = 45° \) to \( \theta_2 = 135° \).

It is easily observed that in all cases, eddy-current contours follow the shape of the excitation as it is expected. The \( r_0 \)-excitation has the best coupling with the conducting sphere from all other excitations. This is the reason why the amplitude of the induced eddy currents is the highest.

The latitudinal component of the eddy-current density of the \( \theta_0 \)-excitation has the same pattern and amplitude as the longitudinal one of the \( \phi_0 \)-excitation, and this also holds for the longitudinal component of the \( \theta_0 \)-excitation and the latitudinal one of the \( \phi_0 \)-excitation. This is attributed to the fact that the two excitations have the same dimensions with the same position with respect to the sphere and the one results from the other by a 90° rotation.

The numerical treatment was tested against the analytical one for the \( r_0 \)-excitation. The FDM scheme, described in the previous paragraph, is used to calculate the SOVP and the eddy currents induced in the sphere. The particular combination of the problem parameters results in a penetration depth comparable with the radius of the sphere. Thus, the electromagnetic field penetrates throughout the sphere and the whole of its volume has to be discretized. Various grids have been used, each having a uniform spatial step variation toward the radial, the latitudinal, and the longitudinal directions. In all cases, the whole discretized area is a sphere subdivided into nonrectangular hexahedra except at the polar axis where they become wedges. The outer boundary of this spherical area should be far enough to approximate infinity. Preliminary analysis showed that we can set the outer boundary at a radius of 0.4 m since, at this distance, the potential is very close to zero. For the particular problem studied, convergence with the SOR is very rapidly attained. The magnetic vector potential and, therefore, the eddy currents are computed numerically from the node values of the SOVP by discretizing (18).

A comparison between the analytical and numerical treatments is done on the basis of the mean square error \( E \), defined as

\[
E = \frac{\sum_{i=1}^{N} |W_{\text{numerical}} - W_{\text{analytical}}|^2}{\sum_{i=1}^{N} W_{\text{analytical}}^2} \times 100\% \tag{48}
\]

where \( N \) is the total number of nodes in the sphere where \( W_{\text{analytical}} \) is calculated. The results are summarized in Table I, from which it is easily observed that the finer the discretization, the greater the accuracy. Another case that is studied numerically involves a rectangular excitation, depicted in Fig. 8. This particular excitation is located on the plane \( y = 0, x \geq 0 \), in the Cartesian coordinate system which corresponds to the \( \phi_0 = 90° \) plane in the spherical coordinate system and is named here...
Fig. 9. Amplitude contours of eddy-current density on the surface of the sphere, (a) $\theta$ component, (b) $\phi$ component, in [A/m$^2$], presented under a Mercator’s projection. Excitation is of the $y_\alpha$-type and the calculations have been performed numerically.
Another comparison that can be performed is between the two excitations located on the plane of constant latitude, the \( \phi_0 \)- and \( \psi_0 \)-excitations. Although the two excitations produce a similar pattern of eddy currents, the \( \phi_0 \)-excitation produces higher and more distributed eddy currents than the \( \psi_0 \)-excitation as it follows the curvature of the surface and achieves a better coupling with the conducting sphere.

VIII. CONCLUSIONS

In this paper, the eddy-current problem for the case of a conducting sphere, excited by a filamentary excitation of arbitrary shape, is studied. Since the problem is 3-D and a solution based upon the magnetic vector potential presented inherent problems, the SOVP formulation has been used for both the analytical and the numerical treatment.

In particular, analytical solutions were derived for the cases when the excitation was located on one of the three surfaces, defining the spherical system of coordinates. Although exact solutions for these three particular excitation shapes have been given in the form of double series, further work is needed to evaluate analytically the surface integral of (34) in order to generalize the model and solve the eddy-current problem for a variety of excitation shapes.

Instead of trying to perform this analytical evaluation for other excitation shapes, we turned to a numerical treatment of the problem by using the FDM.

In both the analytical and numerical implementations the results were very encouraging showing that the use of the SOVP formulation is a very promising tool for the solution of problems described in spherical coordinates, since it leads to the scalarization of the vector field equation and to considerable savings in computing. A notable result of the imposition of the boundary conditions, in the problem studied, was that the electromagnetic field inside the conducting sphere depends only on a single scalar quantity and the eddy-current density lacks a radial component. This fact led to a significant simplification of the problem studied, resulting in analytical expressions of a compact form and in the reduction of computer burden for the application of the FDM.

REFERENCES


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