Development of a Reduced Dispersion-Error Method for the Efficient Treatment of Time-Dependent Electromagnetic Wave Interactions

Konstantinos S. Charitou, Nikolaos V. Kantartzis, and Christos S. Antonopoulos
Dept. of Electrical and Computer Engineering, Aristotle University of Thessaloniki, GR-54124 Thessaloniki, Greece

Abstract—A reduced dispersion-error algorithm for the solution of partial differential equations (PDEs) that govern the evolution of several electromagnetic problems is introduced in this paper. The derivation of the technique breaks away from the rigid finite-differencing background implemented in most typical time-domain formulations. A transfer function and state space approach is adopted instead, yielding thus the correct time advancement of the state space description. In contrast with customary treatments, no explicit spatial discretization is required, while the resulting schemes are guaranteed to be stable. So, the method developed herein is universally applicable to all linear PDEs and provides mechanisms for the use of supplementary error-minimization concepts.

I. INTRODUCTION

Time domain solutions for partial differential equations (PDEs) describing diverse electromagnetic phenomena are widely utilized due to their simplicity and algorithmic applicability. However, as the demands for accuracy and computational efficiency increase in order to handle problems of ongoing difficulty, the need for more systematic formulations arises. One of the most insistent and cumbersome shortcomings to overcome are the artificial dispersion errors inherent in any numerical simulation [1]. This particular lattice reflection defect has been addressed several times in diverse ways but primarily through powerful higher-order finite-difference methods (FDTD) methods [2-4], via the incorporation of anisotropic material constants in the problem space [5] or by advanced spatial and temporal integration schemes [6].

It is with the preceding issues in mind that we present, from first principles, a novel set of techniques which allow the accurate modeling of time-dependent wave interactions. Based on a completely different and mathematically robust approach, the field PDEs are cast into state space or transfer function form, employing integral transformations for the independent variables. Furthermore, spatial discretization is automatically associated with the time update process to assure the stability of the entire scheme. As a consequence, the proposed method attains a very dense node interconnection, which not only has appreciably enhanced dispersion attributes but also features additional flexibility for surfaces obliquely oriented to the computational grid. These advantageous properties are numerically verified by a variety of 2- and 3-D applications.

II. DERIVATION OF REDUCED DISPERSION-ERROR FORMS

An important aspect of our algorithm and of the development itself is their simplicity and intuitiveness both conceptually and in terms of realization. As a starting point, assume the 2-D wave equation for the electric field component $E_z$:

$$
\frac{\partial^2 E_z}{\partial t^2} - \frac{1}{\mu \varepsilon} \left( \frac{\partial^2 E_z}{\partial x^2} + \frac{\partial^2 E_z}{\partial y^2} \right) + \frac{du}{dt} = 0
$$

where $u$ is an arbitrary (distributed or localized) electric field excitation included for completeness. The first step is to capture (1) with a state space description generically given by:

$$
\dot{x} = Ax + Bu \quad \text{and} \quad y = Cx
$$

with matrices $A$, $B$, $C$ formed by the problem’s structural and constitutive parameters. Considering an infinite space we next use Fourier transform with respect to space and define state variables as $x_1 = E_x$, $\dot{x}_1 = \partial E_x / \partial t$. Then, (1) becomes:

$$
\begin{bmatrix}
0 \\
\Omega_1^2 + \Omega_2^2 \\
\mu \varepsilon
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2 \\
x_3
\end{bmatrix}
+ 
\begin{bmatrix}
0 \\
1 \\
0
\end{bmatrix}
\begin{bmatrix}
\dot{x}_1 \\
x_2 \\
x_3
\end{bmatrix}
+ 
\begin{bmatrix}
1 \\
0
\end{bmatrix}
u,
\begin{bmatrix}
y_1 \\
y_2 \\
y_3
\end{bmatrix} = 
\begin{bmatrix}
y_1 \\
y_2 \\
y_3
\end{bmatrix}
(3)
$$

in which $\Omega$ is the wavelength in the direction normal to the wavefront, while $\Omega_x = \mu \varepsilon \cos \phi$ and $\Omega_y = \mu \varepsilon \sin \phi$ are its projections to mesh axes relative to angle $\phi$. In this manner, the associated transfer function representation related to (1) is acquired via the Laplace transform of (3) with respect to time:

$$
y / u = C(sI - A)^{-1}B = \frac{s}{s^2 + \left( \Omega_x^2 + \Omega_y^2 \right) / \mu \varepsilon}.
$$

The transfer function representation of the PDE (3) does not depend on the number of space dimensions [7]. It merely relates the temporal frequency to a spatial one. The implication is that the transfer function does not in any way define a structure for the grid of the time-domain method. In fact the resulting model assumes that each node is symmetrically interconnected with all of its immediate neighbors. It this important observation which will be exploited to determine an interconnection scheme with reduced dispersion error.

We shall, now, concentrate on the discretization of the transfer function, since it produces an effective algorithm. Take into account the one-dimensional case in order to illustrate the basic steps of the technique. The discretization of (4) is attained via the use of the matched pole zero transform [8] that maps poles and zeros into discrete time as:

$$
s + b = 1 - e^{-jT} z^{-1} \quad \text{with} \quad z = e^{j\omega T}
$$

(5)

The unit delay operator and $T$ the sampling rate. Subsequently, the transfer function zeros at infinity are mapped to $z = 1 + 1$ and:

$$
y / u = \frac{1 - z^2}{1 + z^2 - 2z \cos \left( \frac{\Omega_x^2 + \Omega_y^2}{\mu \varepsilon} T \right)}.
$$

(6)

The time-domain solution is, then, obtained via the inversion of (6) in space and time. Owing to the fact that in two and three dimensions it is not possible to use an analytical inverse Fourier transform for the spatial-frequency expression, an approximate counterpart has to be determined.
It is a common knowledge that the inverse of such a spatial operator defines an interconnection between the nodes. The error involved in the approximation of this inversion gives rise to the dispersion error whose reduction requires an improved interconnection scheme. There are various schemes that can be constructed by means of our approach. A typical 2-D one is shown in Fig. 1. Its dispersion relation reads

$$2\cos\left(N\sqrt{\Omega_1^2 + \Omega_2^2}\right) \approx c_1\left[\cos\left(N\Omega_1\right) + \cos\left(N\Omega_2\right)\right] + c_2\sin\left(N\Omega_1\right)\cos\left(N\Omega_2\right)$$

with $c_2 = 2(1 - c_1), N = T(\mu\epsilon)^{1/2}$ and is four times less sensitive to the variation of the propagation angle than the usual finite-difference method. More sophisticated schemes can also be easily derived through the procedure presented here.

The densely interconnected nature of the proposed algorithm has an additional advantage. As the relevant literature reveals, the staircase approximation involved when modeling oblique boundaries or curved surfaces, which in effect creates parasitic resonance cavities along the boundary, is a significant source of error. Our approach reduces this staircase deficiency during the discretization of such geometries.

III. NUMERICAL RESULTS

A variety of 2- and 3-D problems were considered for the numerical validation of the proposed algorithm. As an indicative example, we focus on the propagation of a Gaussian pulse of duration $1.7368 \times 10^{-11}$ sec was applied at the center of the computational domain. The spatial step was set to $\Delta x = 0.001 \text{ m}$. For the improved interconnection scheme the space time relationship is $\Delta x = 0.95T(\mu\epsilon)^{1/2}$. It should be noted that this relationship does not affect the stability of the solution but only its consistency. The standard FDTD solution together with that based on the interconnection scheme of Fig. 1 are, respectively, shown in Figs 2a and 2b. The new arrangement shows visible improvement compared to the standard FDTD solution. The propagation front is symmetric about the source with consistent amplitude in every direction.

It is to be emphasized that the improved characteristics and properties of our formulation enables its implementation in problems that involve steep excitations. In general, the error associated with these schemes is not always reduced by selecting a seriously decreased time-step. The proposed method however accomplishes considerable reductions in such cases, utilizing a temporal increment larger than the maximum allowable analogue of the conventional FDTD methods.

IV. CONCLUSIONS

A set of robust methods has been derived using a rigorous and consistent approach. The resulting algorithms exhibit reduced dispersion error characteristics. The densely interconnected nature of the schemes produced by the approach adopted here provides greater flexibility in modeling complex structures with improved precision. Finally, the concepts of transfer functions and the methods related to their analysis have been introduced in numerical electromagnetics.

REFERENCES


